Harmonious Many-Valued Propositional Logics and the Logic of Computer Networks

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ABSTRACT. In this paper we reconsider the notion of an n-valued propositional logic. In many-valued logic, sometimes a distinction is made not only between designated and undesignated (not designated) truth values, but between designated, undesignated, and antidesignated truth values. But even if the set of truth values is, in fact, tripartitioned, usually only a single semantic consequence relation is defined that preserves the possession of a designated value from the premises to the conclusions of an inference. We shall argue that if the set of semantical values is not bipartitioned into the designated and the antidesignated truth values, it is natural to define *two* entailment relations, a positive one that preserves possessing a designated value from the premises to the conclusions of an inference, and a negative one that preserves possessing an antidesignated value from the conclusions to the premises. Once this distinction has been drawn, it is quite natural to reflect it in the logical object language and to contemplate many-valued logics Λ, whose language is split into a positive and a matching negative logical vocabulary. If the positive and the negative entailment relations do not coincide, the interpretations of matching pairs of connectives are distinct, and nevertheless the positive entailment relation restricted to the positive vocabulary is isomorphic to the negative entailment relation restricted to the negative vocabulary, then we shall say that Λ is a *harmonious* many-valued logic. We shall present examples of harmonious finitely-valued logics. These examples are not *ad hoc*, but emerge naturally in the context of generalizing Nuel Belnap's ideas on how a single computer should think to how interconnected computers should reason. We shall conclude this paper with some remarks on generalizing the notion of a harmonious n-valued propositional logic.

1 Many-valued propositional logics generalized

There exists a simple definition of the notion of an extensional n-valued ($2 \leq n \in \mathbb{N}$) propositional logic as a valuational system, see, for example, [22] or the standard definition of a logical matrix, say in [18]. According to this definition, an n-valued propositional logic is a structure

$$\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in C\} \rangle,$$

where $\mathcal{V}$ is a non-empty set containing n elements ($2 \leq n$), $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, C is the (non-empty, finite) set of (primitive) connectives of some propositional language $\mathcal{L}$, and every f_c is a function on $\mathcal{V}$ with the same arity as c. The elements of $\mathcal{V}$ are usually called *truth values*, and the elements of $\mathcal{D}$ are regarded as the *designated* truth values. A structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in C\}\rangle$ may be viewed as a logic, because the set of designated truth values determines a relation of semantical consequence (entailment) $\models \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$, where $\mathcal{P}(\mathcal{L})$ is the powerset of $\mathcal{L}$. A valuation function v is a function from the set of all atomic formulas (alias sentence letters) into $\mathcal{V}$. Every valuation function v is inductively extended to a function from the set of all $\mathcal{L}$-formulas into $\mathcal{V}$ by the following definition:

$$v(c(A_1, \ldots, A_m)) \quad = \quad f_c(v(A_1), \ldots, v(A_m)),$$

where c is an m-place connective from C. A set of formulas Δ *entails* a set of formulas Γ ($\Delta \models \Gamma$) iff for every valuation function v the following holds true: if for every $A \in \Delta$, $v(A) \in \mathcal{D}$, then $v(B) \in \mathcal{D}$ for some $B \in \Gamma$. An n-valued tautology then is a formula A such that $\varnothing \models A$.

In some writings, the definition of an n-valued propositional logic and the terminology is slightly different. First, the elements of $\mathcal{V}$ are sometimes referred to as *quasi truth values*. Gottwald [14, p. 2] explains that one reason for using the term 'quasi truth value' is that there is no convincing and uniform interpretation[1] of the truth values that in many-valued logic are taken in addition to the classical truth values *true* and *false*, an understanding that, according to Gottwald, associates the additional values with the naive understanding of being true, respectively the naive understanding of degrees of being true. In later publications, Gottwald has changed his terminology and states that "to [a]void any confusion with the case of classical logic one prefers in many-valued logic to speak of *truth degrees* and to use the word "truth value" only for classical logic" [15, p. 4]. The term 'semantical value' (or just 'value') seems to be non-committal.

What is perhaps more important than these differences in terminology is that in part of the literature, for example in [14], [15], [18], [23], an explicit distinction is drawn between a set $\mathcal{D}^+$ of *designated* values and a set $\mathcal{D}^-$ of *antidesignated* values, where the latter need not coincide with the complement of $\mathcal{D}^+$.[2] Usually, this distinction is, however, not fully exploited in many-valued logic: The notion of entailment is defined with respect to the designated values, and no independent additional entailment relation is defined with respect to the antidesignated values.

[1] At least there was no such interpretation at the time of the writing of [14].

[2] Incidentally, this distinction is relevant for an assessment of Suszko's Thesis, see [32], the claim that "there are but two logical values, true and false" [6, p. 169], which is given a formal content by the so-called Suszko Reduction, the proof that every Tarskian n-valued propositional logic is also characterized by a bivalent semantics. For a recent treatment and references to the literature, see [6]. A critical discussion of Suszko's Thesis is presented in [36], which is a companion article to the present paper.

Gottwald is quite aware of the distinction between designated and antidesignated values. When he discusses the notion of a contradiction (or logical falsity), for example, he explains that there are two ways of generalizing this notion from classical logic to many-valued logic, [15, p. 32, notation adjusted]:

> 1. In the case that the given system S of propositional many-valued logic has a suitable negation connective $\sim$, one can take as logical falsities all those wffs H for which $\sim H$ is a logical truth.
> 2. In the case that the given system S of propositional many-valued logic has antidesignated truth degrees, one can take as logical falsities all those wffs H which assume only antidesignated truth degrees. . .

Gottwald assumes that $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$ and remarks that if the designated and antidesignated values exhaust all truth degrees and, moreover, the negation operation $\sim$ satisfies the following standard condition (notation adjusted):

$$f_{\sim}(x) \in \mathcal{D}^+ \text{ iff } x \notin \mathcal{D}^+,$$

then the two notions of a contradiction coincide [15, p. 32]. Since $\mathcal{D}^-$ may differ from the complement of $\mathcal{D}^+$, another standard condition for negation $\sim$ is:

$$f_{\sim}(x) \in \mathcal{D}^+ \text{ iff } x \in \mathcal{D}^- \text{ and } f_{\sim}(x) \in \mathcal{D}^- \text{ iff } x \in \mathcal{D}^+.$$

If the latter condition is satisfied, the two ways of defining the notion of a contradiction are equivalent also if $\mathcal{V} \setminus \mathcal{D}^+ \neq \mathcal{D}^-$.

Rescher [23, p. 68], who does not consider semantic consequence but only tautologies and contradictions, explains that "there may be good reason for *letting one and the same truth-value be both designated and antidesignated*." However, he also warns that "we would not want it to happen that there is some truth-value v which is both designated and antidesignated when it is also the case that there is some formula which uniformly assumes this truth-value, for then this formula would be both a tautology and a contradiction" [23, p. 67].

Although Gottwald recognizes that $\mathcal{V} \setminus \mathcal{D}^+$ may be distinct from $\mathcal{D}^-$,[3] he nevertheless follows the tradition in defining only a single semantic consequence rela-

[3]On p. 30 of [15] he explains (notation adjusted):

> Even in the case that $\mathcal{D}^+ \neq \varnothing$ and $\mathcal{D}^- \neq \varnothing$ it is, however, not necessarily $\mathcal{D}^+ \cup \mathcal{D}^- = \mathcal{V}$, which means that together with designated and antidesignated truth degrees also *undesignated* truth degrees may exist. This possibility indicates two essentially different positions regarding the designation of truth degrees. The first one assumes only a binary division of the set of truth degrees and can proceed by simply marking a set of designated truth degrees, treating the undesignated ones like antidesignated ones. The second position assumes a tripartition and marks some truth degrees as designated, some others as antidesignated, and has besides these both types also some undesignated truth degrees. . . .

tion in terms of $\mathcal{D}^+$. If this privileged treatment of $\mathcal{D}^+$ is given up, a more general definition of an n-valued propositional logic emerges.

DEFINITION 1. Let $\mathcal{L}$ be a language in a denumerable set of sentence letters and a finite non-empty set of finitary connectives C. An n-valued propositional logic is a structure

$$\langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in C\}\rangle$$

where $\mathcal{V}$ is a non-empty set containing n elements ($2 \leq n$), $\mathcal{D}^+$ and $\mathcal{D}^-$ are distinct non-empty proper subsets of $\mathcal{V}$, and every f_c is a function on $\mathcal{V}$ with the same arity as c. Again, every valuation v is inductively extended to a function from the set of all $\mathcal{L}$-formulas into $\mathcal{V}$ by setting: $v(c(A_1, \ldots, A_m)) = f_c(v(A_1), \ldots, v(A_m))$, for every m-place $c \in C$. For all sets of $\mathcal{L}$-formulas Δ, Γ, semantic consequence relations $\models^+$ and $\models^-$ are defined as follows:

1. $\Delta \models^+ \Gamma$ iff for every valuation function v: (if for every $A \in \Delta$, $v(A) \in \mathcal{D}^+$, then $v(B) \in \mathcal{D}^+$ for some $B \in \Gamma$);

2. $\Delta \models^- \Gamma$ iff for every valuation function v: (if for every $A \in \Gamma$, $v(A) \in \mathcal{D}^-$, then $v(B) \in \mathcal{D}^-$ for some $B \in \Delta$).

An n-valued tautology then is a formula A such that $\emptyset \models^+ A$, and an n-valued contradiction is a formula A such that $A \models^- \emptyset$.

DEFINITION 2. Let $\Lambda = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in C\}\rangle$ be an n-valued propositional logic. Λ is called a *separated* n-valued logic, if $\mathcal{V} \setminus \mathcal{D}^+ \neq \mathcal{D}^-$ (that is if $\mathcal{V}$ is not partitioned into the non-empty sets $\mathcal{D}^+$ and $\mathcal{D}^-$), and Λ is said to be *refined*, if it is separated and $\models^+ \neq \models^-$.

Clearly, if an n-valued logic is not separated, the two entailment relations $\models^+$ and $\models^-$ coincide. In a refined n-valued propositional logic, however, neither "positive" entailment $\models^+$ nor "negative" entailment $\models^-$ need enjoy a privileged status in comparison to each other. Moreover, an entailment relation is usually defined with respect to a given formal language. For the separate entailment relations $\models^+$ and $\models^-$ one might, therefore, expect that they come with their own, in a sense "dual", languages, $\mathcal{L}^+$ and $\mathcal{L}^-$ (see the next section for further explanations). Then one might be interested in n-valued logics in which these languages have the same signature.

DEFINITION 3. Let C be a finite non-empty set of finitary connectives, let $\mathcal{L}^+$ be the language based on $C^+ = \{c^+ \mid c \in C\}$, and let $\mathcal{L}^-$ be the language based on $C^- = \{c^- \mid c \in C\}$. If A is an $\mathcal{L}^+$-formula, let A^- be the result of replacing every connective c^+ in A by c^-. If Δ is a set of $\mathcal{L}^+$ formulas, let $\Delta^- = \{A^- \mid A \in \Delta\}$. If the language $\mathcal{L}$ of a refined n-valued logic Λ is based on $C^+ \cup C^-$, then Λ is said to be *harmonious* iff (i) for all sets of $\mathcal{L}^+$-formulas Δ, Γ: $\Delta \models^+ \Gamma$ iff $\Delta^- \models^- \Gamma^-$, and (ii) for every $c \in C$, $f_{c^+} \neq f_{c^-}$.

In the present paper, we shall, first of all, argue that the distinction between designated and antidesignated values, and therefore also the distinction between positive entailment $\models^+$ and negative entailment $\models^-$, is an important distinction (Section 2). In Section 3 we shall consider some separated n-valued propositional logics, and in Sections 4 and 5 we shall present natural examples of harmonious finitely-valued logics. Finally, we shall make some brief remarks on generalizing harmony (Section 6).

2 Designated and antidesignated values

Why is it important to draw a distinction between designated and antidesignated values? The notion of a set of designated values is often considered as a generalization of the notion of truth. Similarly the set of antidesignated values can be regarded as representing a generalized concept of falsity. However, logic and its terminology is to a large extent predominated by the notion of truth. The Fregean *Bedeutung* of a declarative sentence (or, in the first place, a thought) is a truth value. According to Frege, there are exactly two truth values, *The True* and *The False*, henceforth just *true* (T) and *false* (F). Whereas T and F are referred to as *truth* values, neither F nor T is called a *falsity* value.

Moreover, Frege explicitly characterized logic as "the science of the most general laws of being true", see H. Sluga's translation in [31, p. 86]. This view finds its manifestation in the fact that most of the fundamental logical notions – logical operations, relations etc. – are usually defined through the category of truth. Thus, valid consequence is usually defined as preserving truth in passing from the premises to the conclusions of an inference. It is required that for every model $\mathfrak{M}$, if all the premises are true in $\mathfrak{M}$, then so is at least one conclusion. By contraposition, in a valid inference being not true is preserved from the conclusions to at least one of the premises. For every model $\mathfrak{M}$, if every conclusion is not true in $\mathfrak{M}$, then so is at least one of the premises. As to falsity, its role in such definitions frequently remains a subordinated one, if any. When constructing a semantic model, "false" is often understood as a mere abbreviation for "not true", for instance when the classical truth-table definition for conjunction is stated as: "A conjunctive sentence is true if both of its conjuncts are true, otherwise it is not true (i.e., false)."

In general failing to be true and being false may, however, fall apart. Although this is the very point of many-valued logic, a distinction between falsity and the absence of truth (or truth and the absence of falsity) is often represented only by the values that $\mathcal{V}$ contains in addition to T and F, and not by distinguishing between a set of designated values $\mathcal{D}^+$ and another set of antidesignated values $\mathcal{D}^-$ (and the consequence relations induced by these sets). We are interested in inferences that preserve truth, because we are interested in true beliefs. But likewise we are interested in inferences in which false conclusions are bound to depend on at least one false premise, because we are interested in avoiding false beliefs. This point has vividly been made by William James in his essay 'The will to believe':

> Believe truth! Shun error!—These, we see, are two materially different laws; and by choosing between them we may end by coloring differently our whole intellectual life. We may regard the chase for truth as paramount, and the avoidance of error as secondary; or we may, on the other hand, treat the avoidance of error as more imperative, and let truth take its chance [16, p. 18].

It seems then quite natural to modify (or to extend) in a certain respect the Fregean definition of logic by saying that its scope is studying not just the laws of being true, but rather of being true *and* being false. An immediate effect of this modification consists in acknowledging the importance of falsity (and more generally, antidesignated values) for defining logical notions. Every definition formulated in terms of truth should be "counterbalanced" with a "parallel" (dual) definition formulated in terms of falsity.

This observation not only justifies the distinction between positive ($\models^+$) and negative ($\models^-$) entailment relations, but also clarifies the idea of the corresponding languages $\mathcal{L}^+$ and $\mathcal{L}^-$ with "positive" and "negative" connectives. As an example let us take the operation of conjunction in classical logic. The connective $\wedge^+$ can naturally be defined through its truth conditions: $v(A \wedge^+ B) = T$ iff $v(A) = T$ and $v(B) = T$. But we may also wish to consider the connective $\wedge^-$, exhaustively defined by means of the *falsity conditions*: $v(A \wedge^- B) = F$ iff $v(A) = F$ or $v(B) = F$. Now, although in classical logic $\wedge^+$ and $\wedge^-$ are, obviously, coincident, in the general case, for example in various many-valued logics, they may well differ from each other. Thus, if the relation between designated and antidesignated values is not so straightforward as in classical logic, a separate introduction of the positive and negative connectives (in fact: "truth-connectives" and "falsity-connectives") may acquire especial significance.

Let us generalize this point with respect to the propositional connectives of conjunction and disjunction in the context of many-valued logics. There is a view that in a many-valued semantics $f_\wedge$ and $f_\vee$ are just the functions of taking the minimum and the maximum of their arguments. As R. Dewitt put it:

> In many-valued systems, intuitions concerning the appropriate truth-conditions for disjunction and conjunction are the most widely agreed on. In particular, there is general agreement that a disjunction should take the maximum value of the disjuncts, while a conjunction should take the minimum value of the conjuncts [8, p. 552].

As a result, we get the following definitions:

$$\begin{aligned} v(A \wedge B) &= min(v(A), v(B)); \\ v(A \vee B) &= max(v(A), v(B)), \end{aligned}$$

which Dewitt refers to as the *standard conditions* for conjunction and disjunction. One may note, however, that these conditions are justified only if the set of semantical values is in some way *linearly ordered*, so that any two values are mutually comparable. And although this is frequently indeed the case, e.g., when the semantical values are identified with some points on a numerical segment, there also exist a number of many-valued systems where not all of the values are comparable with each other.[4] In such systems, the standard conditions cannot be employed directly.

The idea of truth *degrees* in many-valued logic naturally implies that semantical values may differ in their truth-content. Moreover, it is assumed that any of the designated values is "more true" (is of a higher degree in its truth-content) than any of the values which is not designated. Thus, taking into account the "minimality-maximality" intuition described above, conjunction can be more generally regarded as an operation $\wedge^+$ that in a sense *minimizes the truth-content* (or the "designatedness") of the conjuncts, and disjunction as an operation $\vee^+$ that *maximizes the truth-content* of the disjuncts.[5] That is, for truth values that are comparable in their truth degrees, $f_{\wedge^+}$ is just the standard *min*-function, but if two truth values x and y are incomparable, the "less true"–relation nevertheless should be such that it determines $f_{\wedge^+}(x, y)$, the outcome of which is *less true* than *both* of the conjuncts. And similarly for disjunction. In this way, we should be able to obtain definitions of notions of conjunction $\wedge^+$ and disjunction $\vee^+$ purely in terms of truth:

$$\begin{aligned} v(A \wedge^+ B) &= min^+(v(A), v(B)); \\ v(A \vee^+ B) &= max^+(v(A), v(B)), \end{aligned}$$

where min^+ and max^+ are generalized functions of truth-minimizing and truth-maximizing, correspondingly.

Now, if falsity is not the same as non-truth, an independent consideration of propositional connectives from the standpoint of antidesignated values is appropriate. In this sense conjunction should be regarded as the falsity-maximizer and disjunction as the falsity-minimizer, and thus, we should be able to obtain generalized functions max^- and min^- of maximizing and minimizing, respectively, the falsity-content of their arguments, so that operations $\wedge^-$ and $\vee^-$ can be defined purely in terms of falsity:

$$\begin{aligned} v(A \wedge^- B) &= max^-(v(A), v(B)); \\ v(A \vee^- B) &= min^-(v(A), v(B)). \end{aligned}$$

We also briefly observe the difference between two kinds of negation . Namely, whereas $f_{\sim^+}$ can be viewed as a function that (within every semantical value) turns

[4] Cf., for instance, the values **N** and **B** under the order $\leq_t$ in the four-valued logic B_4 considered in Section 4.

[5] And of course, a conjunction should maximize the non-truth of the conjuncts, while a disjunction should minimize the non-truth of the disjuncts.

truth into non-truth and vice versa, $f_{\sim^-}$ can be treated as an operation that interchanges exclusively between falsity and non-falsity.

One might object that the distinction between designated values and antidesignated values makes sense *only* for doxastically or epistemically interpreted semantical values. Certainly, if a proposition is not believed (known) to be true, this does not imply that the proposition is believed (known) to be false. The distinction is, however, sensible also for other, non-doxastical and non-epistemical adverbial qualifications of truth and falsity. If a proposition is not necessarily true, for instance, it need not be necessarily false. The designated semantical values are used to define an entailment relation $\models^+$ that preserves possessing a doxastically wanted value in passing from the premises to the conclusions of an inference. Analogously, the antidesignated values may be used to define an entailment relation $\models^-$ by requiring that if the conclusions are doxastically unwanted, at least one of the assumptions is doxastically unwanted, too. Among the doxastically wanted values there may be values interpreted, for example, as "true", "neither true nor false", "known to be true", "unknown to be false", "necessarily true", "possibly true", etc. Among the doxastically unwanted values there may be values interpreted, for example, as "false", "neither true nor false", "known to be false", "unknown to be true", "necessarily false", "possibly false", etc.

Whether "neither true nor false" is doxastically wanted or unwanted may be a matter of perspective. A proposition evaluated as "neither true nor false" is not falsified and hence possibly wanted, but it is also not verified and therefore possibly unwanted. If one wants to take into account that both perspectives are legitimate, a value read as "neither true nor false" may even sensibly be classified as both wanted and unwanted. That is, in general it is reasonable to distinguish between designated, antidesignated, and undesignated semantical values, and also not to exclude the possibility of values that are both designated and antidesignated.

These considerations can be generalized, see Section 6.

3 Some separated finitely-valued logics

We first consider a separated version of Kleene's well-known 3-valued logic.

DEFINITION 4. Kleene's strong 3-valued logic K_3 is defined as follows:

$K_3 = \langle\{T, \emptyset, F\}, \{T\}, \{\emptyset, F\}, \{f_c : c \in \{\sim, \wedge, \vee, \supset\}\}\rangle$, where the functions f_c are defined by the following tables:

$f_\sim$	
T	F
$\emptyset$	$\emptyset$
F	T

$f_\wedge$	T	$\emptyset$	F
T	T	$\emptyset$	F
$\emptyset$	$\emptyset$	$\emptyset$	F
F	F	F	F

$f_\vee$	T	$\emptyset$	F
T	T	T	T
$\emptyset$	T	$\emptyset$	$\emptyset$
F	T	$\emptyset$	F

$f_\supset$	T	$\emptyset$	F
T	T	$\emptyset$	F
$\emptyset$	T	$\emptyset$	$\emptyset$
F	T	T	T

DEFINITION 5. The separated 3-valued propositional logic K_3* is defined as follows:

$K_3* := \langle \{T, \emptyset, F\}, \{T\}, \{F\}, \{f_c : c \in \{\sim, \wedge, \vee, \supset\}\}\rangle$, where the functions f_c are defined as in K_3.

OBSERVATION 6. K_3* is refined, i.e., the relations $\models^+$ and $\models^-$ do not coincide.

Proof. In K_3* formulas of the form $A \wedge (A \supset B)$ have the following truth table:

	A	B	$A \wedge (A \supset B)$
	T	T	T
	T	$\emptyset$	$\emptyset$
	T	F	F
	$\emptyset$	T	$\emptyset$
	$\emptyset$	$\emptyset$	$\emptyset$
*	$\emptyset$	F	$\emptyset$
	F	T	F
	F	$\emptyset$	F
	F	F	F

Whereas $A \wedge (A \supset B) \models^+ B$, the row marked with an asterisk shows that $A \wedge (A \supset B) \not\models^- B$. ■

Although it is not surprising, perhaps, that in a separated n-valued logic the relations $\models^+$ and $\models^-$ need not coincide, there are separated n-valued logics which are not refined. Let $\mathbf{N} := \emptyset$, $\mathbf{T} := \{T\}$, $\mathbf{F} := \{F\}$ and $\mathbf{B} := \{T, F\}$.

DEFINITION 7. The useful 4-valued logic of Dunn and Belnap is the structure $B_4 = \langle \{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{\mathbf{N}, \mathbf{F}\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$, where the functions f_c are defined as follows:

$f_\sim$	
T	**F**
B	**B**
N	**N**
F	**T**

$f_\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

$f_\vee$	**T**	**B**	**N**	**F**
T	**T**	**T**	**T**	**T**
B	**T**	**B**	**T**	**B**
N	**T**	**T**	**N**	**N**
F	**T**	**B**	**N**	**F**

DEFINITION 8. The separated 4-valued propositional logic B_4* is the structure $\langle \{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{\mathbf{F}, \mathbf{B}\}, \{f_c : c \in \{\sim, \wedge, \vee, \}\}\rangle$, where the functions f_c are defined as in B_4.

The set $\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}$ is also referred to as $\mathbf{4}$. Note that in B_4* not only $\mathbf{4} \setminus \mathcal{D}^+ \neq \mathcal{D}^-$, but also $\mathcal{D}^+ \cap \mathcal{D}^- \neq \emptyset$.

OBSERVATION 9. The separated logic B_4* is not refined: $\models^+ = \models^-$.

Proof. A proof (for the case of single premises and conclusions) using Dunn's method of "dual" valuations is given, e.g., in [10, p. 10]. There it is observed that the relation $\models^+$ alias $\models^-$ also coincides with the relation $\models$ defined as follows: $\Delta \models \Gamma$ iff ($\Delta \models^+ \Gamma$ and $\Delta \models^- \Gamma$). ■

Clearly, K_3 does not have any tautologies, because any formula takes the value $\varnothing$ if every propositional variable occurring in it takes the value $\varnothing$. It has been observed in [23] that for the same reason the 3-valued logic which is now known as the Logic of Paradox [21], *LP*, has no contradictions.

DEFINITION 10. The Logic of Paradox *LP* is the 3-valued propositional logic $\langle\{T,\varnothing,F\},\{T,\varnothing\},\{F\},\{f_c : c \in \{\sim,\wedge,\vee,\supset\}\}\rangle$, where the functions f_c are defined as in K_3.

In B_4 there are neither any tautologies (consider the constant valuation that assigns only **N**) nor any contradictions (consider the constant valuation that assigns only **B**).[6] Obviously, $\models^+$ in B_4 coincides with $\models^+$ and $\models^-$ in B_4*. Our main question is: Are there *natural* examples of harmonious finitely-valued logics?

4 A harmonious finitely-valued logic

We shall present a harmonious finitely-valued logic that emerges naturally in the context of generalizing Nuel Belnap's ideas on how a computer should reason.

The bilattice $FOUR_2$

Our starting point is the logic B_4, which is also known as the logic of *first degree entailment*. Belnap ([3], [4], see also [1, §81]), building on ideas developed by M. Dunn (see, e.g., [9] and [10]), proposed B_4 as a "useful four-valued logic" for dealing with information received by a computer. As Belnap points out, a computer may receive data from *various* (maybe independent) sources. Belnap's computers have to take into account various kinds of information concerning a given sentence. Besides the standard (classical) cases, when a computer obtains information either that the sentence is (1) true or that it is (2) false, two other (non-standard) situations are possible: (3) nothing is told about the sentence or (4) the sources supply inconsistent information, information that the sentence is true and information that it is false. Thus, we obtain the four truth values from B_4 that naturally correspond to these four "informational" situations: **N** (there is no information that the sentence is false and no information that it is true), **F** (there is *merely* information that the sentence is false), **T** (there is *merely* information that

[6] Rescher [23, p. 67] seems to interpret the fact that the logic *LP* has no contradictions as a reason for distinguishing between antidesignated and undesignated values, because in *LP* no formula receives an undesignated truth value under any valuation.

the sentence is true), and **B** (there is information that the sentence is false, but there is also information that it is true).

M. Ginsberg [12], [13] noticed that the four truth values of B_4 constitute an interesting algebraic structure, which he called a *bilattice*.[7] Figure 1 presents this bilattice ($FOUR_2$) by a double Hasse diagram. We have here two partial orderings:

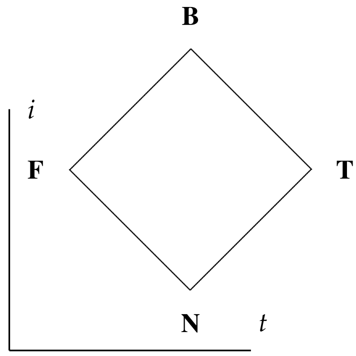

Figure 1. The bilattice $FOUR_2$

$\leq_i$ and $\leq_t$. The relation $\leq_i$ orders the elements of **4** by set-inclusion and represents an increase of information. The relation $\leq_t$ is said to represent an increase of truth among the elements constituting **4**.[8] The relation $\leq_t$ is of special importance, because this order determines the *logic* of Belnap's computers. First, conjunction and disjunction are interpreted as the operations of lattice meet and join relative to $\leq_t$. These operations are given by the truth functions $f_\wedge$ and $f_\vee$ from Definition 7. Moreover, negation $\sim$ is interpreted as the function $f_\sim$ from Definition 7, which represents an inversion of $\leq_t$ in the sense that $x \leq_t y$ iff $f_\sim(y) \leq_t f_\sim(x)$.

A valuation function into **4** is then recursively extended in the standard way to a map v from the language of B_4 into **4**, and the entailment relation of B_4 can be defined as follows:

$$\Delta \models \Gamma \text{ iff } \forall v \bigsqcap_t \{v(A) \mid A \in \Delta\} \leq_t \bigsqcup_t \{v(A) \mid A \in \Gamma\},$$

[7]Roughly speaking a bilattice is a set with *two* partial orderings, each determining its own lattice on this set (see also [2], [11]).

[8]Correspondingly one can distinguish between an information (approximation) lattice and a logical lattice. Belnap [3] has considered both these lattices separately under the labels **A4** and **L4**.

where $\sqcap_t$ is lattice meet and $\sqcup_t$ is lattice join in the complete lattice $(\mathbf{4}, \leq_t)$.[9]

From isolated computers to computer networks

One can observe that Belnap's interpretation works perfectly well if we deal with *one* (isolated) computer receiving information from *classical sources*, i.e., these sources operate exclusively with the classical truth values. As soon as a computer C is connected to other computers, there is no reason to assume that these computers cannot pass higher-level information concerning a given proposition to C. If several computers form a computer network, Belnap's ideas that motivated B_4 can be generalized. Consider, for example, four computers: C_1, C_2, C_3, and C_4 connected to another computer C'_1, a server, to which they are supposed to supply information (Figure 2). It is fairly clear that the logic of the server itself (so, the network as a whole) cannot remain four-valued any more. Indeed, suppose C_1 informs C'_1 that a sentence is true only (has the value **T**), whereas C_2 supplies inconsistent information (the sentence is both true and false, i.e., has the value **B**). In this situation C'_1 has received the information that the sentence simultaneously

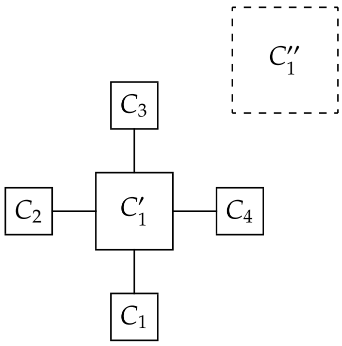

Figure 2. A computer network

is true only as well as both true and false, in other words, it has a value not from **4**, but from $\mathcal{P}(\mathbf{4})$, namely the value $\mathbf{TB} = \{\{T\}, \{T, B\}\}$. If C'_1 has been informed simultaneously by C_1 that a sentence is true-only, by C_2 that it is false-only, by C_3 that it is both-true-and-false, and by C_4 that it is neither-true-nor-false, then the value $\mathbf{NFTB} = \{\varnothing, \{T\}, \{F\}, \{T, F\}\}$ is far from being a "madness" (cf. [20, p. 19]) but is just an adequate value which should be ascribed to the sentence by C'. That

[9] Another standard definition of multiple-conclusion semantic consequence builds compactness into this notion by requiring that $\Delta \models \Gamma$ iff there exist finite subsets $\Delta' \subseteq \Delta$ and $\Gamma' \subseteq \Gamma$ such that for every valuation v, $v(\bigwedge_t \Delta') \leq_t v(\bigvee_t \Gamma')$.

is, the logic of C_1' has to be 16-valued. And if we wish to extend our network and to connect C_1' to some "higher" computer (C_1''), then the amount of semantical values will increase to $2^{16} = 65536$. As we shall see later, this exponential growth of the number of truth values need not worry us too much, since we always will end up with the same semantical (and syntactical) consequence relations, see [29], [30], and Section 5.

Thus, generalizing Belnap's idea of truth values encoding information passed to a computer leads us *in a first step*, when we assume that a computer informed by a classical source informs another computer, from **4** to **16** = $\mathcal{P}(\mathbf{4})$ with the following generalized truth values (where **A** = **NFTB** stands for "all"):

1. $\mathbf{N} = \emptyset$	9. $\mathbf{FT} = \{\{F\}, \{T\}\}$
2. $\boldsymbol{N} = \{\emptyset\}$	10. $\mathbf{FB} = \{\{F\}, \{F, T\}\}$
3. $\boldsymbol{F} = \{\{F\}\}$	11. $\mathbf{TB} = \{\{T\}, \{F, T\}\}$
4. $\boldsymbol{T} = \{\{T\}\}$	12. $\mathbf{NFT} = \{\emptyset, \{F\}, \{T\}\}$
5. $\boldsymbol{B} = \{\{F, T\}\}$	13. $\mathbf{NFB} = \{\emptyset, \{F\}, \{F, T\}\}$
6. $\mathbf{NF} = \{\emptyset, \{F\}\}$	14. $\mathbf{NTB} = \{\emptyset, \{T\}, \{F, T\}\}$
7. $\mathbf{NT} = \{\emptyset, \{T\}\}$	15. $\mathbf{FTB} = \{\{F\}, \{T\}, \{F, T\}\}$
8. $\mathbf{NB} = \{\emptyset, \{F, T\}\}$	16. $\mathbf{A} = \{\emptyset, \{T\}, \{F\}, \{F, T\}\}$.

It appears that this passage from **4** to **16** is essential for obtaining a natural example of a harmonious finitely-valued logic. Whereas in $FOUR_2$ the truth order is defined in terms of both T and F, on the richer set of values **16** separate truth and falsity orderings $\leq_t$ and $\leq_f$ can be isolated. In this 16-valued setting, truth and falsity are thereby treated as independent notions in their own right.

The trilattice $SIXTEEN_3$

According to the truth order of $FOUR_2$, the value **B** is less true than **T**, but this means that the *truth* order takes into account the absence of *falsity*, F, and thus, is in fact a *truth-and-falsity* order.[10] The truth values in **16**, however, allow one to define separately a truth order $\leq_t$ by referring only to the classical value T and a falsity order $\leq_f$ by referring only to F. For every x in **16** we first define the sets x^t, x^{-t}, x^f, and x^{-f} as follows:

$$x^t := \{y \in x \mid T \in y\}; \quad x^{-t} := \{y \in x \mid T \notin y\};$$
$$x^f := \{y \in x \mid F \in y\}; \quad x^{-f} := \{y \in x \mid F \notin y\}.$$

[10] In $FOUR_2$ maximizing (resp. minimizing) truth means simultaneously minimizing (resp. maximizing) falsity. It is therefore impossible in B_4 to discriminate between $\mathcal{L}^+$ and $\mathcal{L}^-$: for every $c \in C$, $f_{c^+} = f_{c^-}$.

DEFINITION 11. For every x, y in **16**:

1. $x \leq_i y$ iff $x \subseteq y$;
2. $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^{-t} \subseteq x^{-t}$;
3. $x \leq_f y$ iff $x^f \subseteq y^f$ and $y^{-f} \subseteq x^{-f}$.

As a result, we obtain an algebraic structure that combines the three (complete) lattices $(\mathbf{16}, \leq_i)$, $(\mathbf{16}, \leq_t)$, and $(\mathbf{16}, \leq_f)$ into the *trilattice* $SIXTEEN_3 = (\mathbf{16}, \leq_i, \leq_t, \leq_f)$, see [28]. $SIXTEEN_3$ is presented by a triple Hasse diagram in Figure 3 (cf. also Figure 5 in [27]).

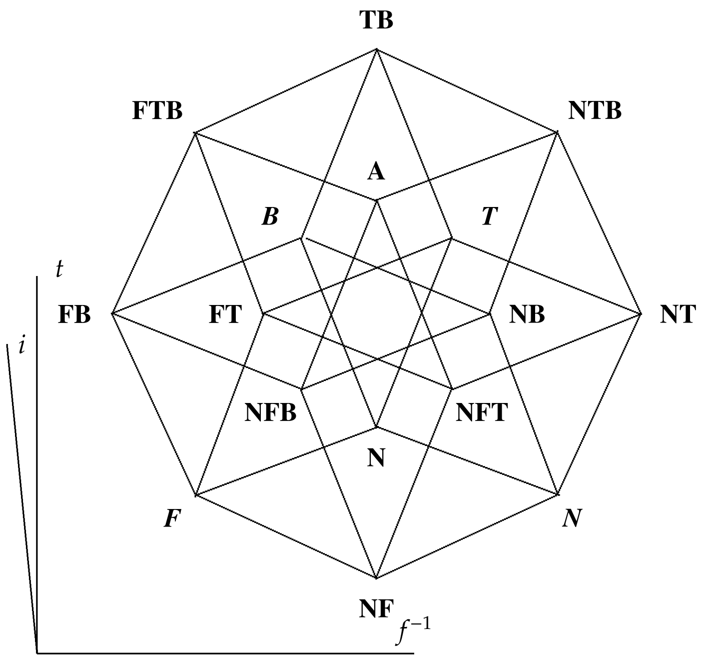

Figure 3. Trilattice $SIXTEEN_3$ (projection $t - f^{-1}$)

Meets and joints exist in $SIXTEEN_3$ for all three partial orders. We will use $\sqcap$ and $\sqcup$ with the appropriate subscripts for these operations under the corresponding ordering relations. Since from the operations one can recover the relations, $SIXTEEN_3$ may also be represented as the structure $(\mathbf{16}, \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f)$. In what follows we will be especially interested in the "logical" operations $\sqcap_t, \sqcup_t, \sqcap_f$ and $\sqcup_f$. Some key properties of these operations are summarized in the following proposition:

PROPOSITION 12. *For any x, y in $SIXTEEN_3$:*

1. $\mathbf{T} \in x \sqcap_t y \Leftrightarrow \mathbf{T} \in x$ *and* $\mathbf{T} \in y$;
   $\mathbf{B} \in x \sqcap_t y \Leftrightarrow \mathbf{B} \in x$ *and* $\mathbf{B} \in y$;
   $\mathbf{F} \in x \sqcap_t y \Leftrightarrow \mathbf{F} \in x$ *or* $\mathbf{F} \in y$;
   $\mathbf{N} \in x \sqcap_t y \Leftrightarrow \mathbf{N} \in x$ *or* $\mathbf{N} \in y$;
2. $\mathbf{T} \in x \sqcup_t y \Leftrightarrow \mathbf{T} \in x$ *or* $\mathbf{T} \in y$;
   $\mathbf{B} \in x \sqcup_t y \Leftrightarrow \mathbf{B} \in x$ *or* $\mathbf{B} \in y$;
   $\mathbf{F} \in x \sqcup_t y \Leftrightarrow \mathbf{F} \in x$ *and* $\mathbf{F} \in y$;
   $\mathbf{N} \in x \sqcup_t y \Leftrightarrow \mathbf{N} \in x$ *and* $\mathbf{N} \in y$;
3. $\mathbf{T} \in x \sqcup_f y \Leftrightarrow \mathbf{T} \in x$ *and* $\mathbf{T} \in y$;
   $\mathbf{N} \in x \sqcup_f y \Leftrightarrow \mathbf{N} \in x$ *and* $\mathbf{N} \in y$;
   $\mathbf{F} \in x \sqcup_f y \Leftrightarrow \mathbf{F} \in x$ *or* $\mathbf{F} \in y$;
   $\mathbf{B} \in x \sqcup_f y \Leftrightarrow \mathbf{B} \in x$ *or* $\mathbf{B} \in y$;
4. $\mathbf{T} \in x \sqcap_f y \Leftrightarrow \mathbf{T} \in x$ *or* $\mathbf{T} \in y$;
   $\mathbf{N} \in x \sqcap_f y \Leftrightarrow \mathbf{N} \in x$ *or* $\mathbf{N} \in y$;
   $\mathbf{F} \in x \sqcap_f y \Leftrightarrow \mathbf{F} \in x$ *and* $\mathbf{F} \in y$;
   $\mathbf{B} \in x \sqcap_f y \Leftrightarrow \mathbf{B} \in x$ *and* $\mathbf{B} \in y$.

Since the relations $\leq_t$ and $\leq_f$ are treated on a par, the operations $\sqcap_t$ and $\sqcup_t$ are not privileged as interpretations of conjunction and disjunction. The operation $\sqcup_f$ may as well be regarded as a conjunction and $\sqcap_f$ as a disjunction. In other words, the logical vocabulary may be naturally split into a positive truth vocabulary and a negative falsity vocabulary. Also certain unary operations with natural negation -like properties are available in $SIXTEEN_3$.

DEFINITION 13. A unary operation $-_t$ $(-_f, -_i)$ on $SIXTEEN_3$ is said to be a t-inversion (f-inversion, i-inversion) iff the following conditions are satisfied:

1. *t-inversion*$(-_t)$:
   (a) $a \leq_t b \Rightarrow -_t b \leq_t -_t a$;
   (b) $a \leq_f b \Rightarrow -_t a \leq_f -_t b$;
   (c) $a \leq_i b \Rightarrow -_t a \leq_i -_t b$;
   (d) $-_t -_t a = a$.
2. *f-inversion*$(-_f)$:
   (a) $a \leq_t b \Rightarrow -_f a \leq_t -_f b$;
   (b) $a \leq_f b \Rightarrow -_f b \leq_f -_f a$;
   (c) $a \leq_i b \Rightarrow -_f a \leq_i -_f b$;
   (d) $-_f -_f a = a$.
3. *i-inversion*$(-_i)$:
   (a) $a \leq_t b \Rightarrow -_i a \leq_t -_i b$;
   (b) $a \leq_f b \Rightarrow -_i a \leq_f -_i b$;
   (c) $a \leq_i b \Rightarrow -_i b \leq_i -_i a$;
   (d) $-_i -_i a = a$.

In $SIXTEEN_3$ such operations are definable as shown in Table 1. Both $-_t$ and $-_f$ are natural interpretations for a negation connective. The following proposition highlights some important properties of these operations.

a	$-_t a$	$-_f a$	$-_i a$	a	$-_t a$	$-_f a$	$-_i a$
N	**N**	**N**	**A**	**NB**	**FT**	**FT**	**FT**
N	***T***	***F***	**NFT**	**FB**	**FB**	**NT**	**FB**
F	***B***	***N***	**NFB**	**TB**	**NF**	**TB**	**TB**
T	***N***	***B***	**NTB**	**NFT**	**NTB**	**NFB**	***N***
B	***F***	***T***	**FTB**	**NFB**	**FTB**	**NFT**	***F***
NF	**TB**	**NF**	**NF**	**NTB**	**NFT**	**FTB**	***T***
NT	**NT**	**FB**	**NT**	**FTB**	**NFB**	**NTB**	***B***
FT	**NB**	**NB**	**NB**	**A**	**A**	**A**	**N**

Table 1. Inversions in $SIXTEEN_3$

PROPOSITION 14. *For any x in $SIXTEEN_3$:*

1. $\mathbf{T} \in -_t x \Leftrightarrow \mathbf{N} \in x$;
$\mathbf{N} \in -_t x \Leftrightarrow \mathbf{T} \in x$;
$\mathbf{F} \in -_t x \Leftrightarrow \mathbf{B} \in x$;
$\mathbf{B} \in -_t x \Leftrightarrow \mathbf{F} \in x$;

2. $\mathbf{T} \in -_f x \Leftrightarrow \mathbf{B} \in x$;
$\mathbf{B} \in -_f x \Leftrightarrow \mathbf{T} \in x$;
$\mathbf{F} \in -_f x \Leftrightarrow \mathbf{N} \in x$;
$\mathbf{N} \in -_f x \Leftrightarrow \mathbf{F} \in x$.

The requirements that the information order $\leq_i$ is left untouched by the operations of t-inversion and f-inversion and that t-inversion (f-inversion) has no effect on $\leq_f$ ($\leq_t$) *are* satisfied by the operations $-_t$ and $-_f$ defined in Table 1, but these requirements might also be given up. If they are abandoned, the definition of t-inversion (f-inversion) refers only to the truth-order (falsity-order). What this suggests is that not only conjunction and disjunction, but also negation emerges in two versions. Moreover, since $x \sqcap_t y \neq x \sqcup_f y$, $x \sqcup_t y \neq x \sqcap_f y$ and $-_t x \neq -_f x$, the two logical orderings $\leq_t$ and $\leq_f$ indeed give rise to two distinct sets of logical operations of the same arity.

A harmonious logic inspired by the logic of $SIXTEEN_3$

An appropriate syntax for the logic emerging from $SIXTEEN_3$ is given by a denumerable set of propositional variables and three propositional languages $\mathcal{L}_t$, $\mathcal{L}_f$, and $\mathcal{L}_{tf}$ based on this set. They are defined in Backus–Naur form as follows:

$$\mathcal{L}_t: \quad A ::= \quad p \mid {\sim_t} A \mid A \wedge_t A \mid A \vee_t A$$
$$\mathcal{L}_f: \quad A ::= \quad p \mid {\sim_f} A \mid A \wedge_f A \mid A \vee_f A$$
$$\mathcal{L}_{tf}: \quad A ::= \quad p \mid {\sim_t} A \mid {\sim_f} A \mid A \wedge_t A \mid A \vee_t A \mid A \wedge_f A \mid A \vee_f A$$

The logic of $SIXTEEN_3$ is semantically presented as a *bi-consequence system*, namely the structure $(\mathcal{L}_{tf}, \models_t, \models_f)$, where the two entailment relations $\models_t$ and $\models_f$

are defined with respect to the truth order $\leq_t$ and the falsity order $\leq_f$, respectively. The main point of the present paper consists in defining the notion of a harmonious finitely-valued propositional logic and pointing out that the logic of $SIXTEEN_3$ and similar structures lead to examples of harmonious many-valued logics. We first define $(\mathcal{L}_{tf}, \models_t, \models_f)$.

DEFINITION 15. Let v be a map from the set of propositional variables into **16**. The function v is recursively extended to a function from the set of all $\mathcal{L}_{tf}$-formulas into **16** as follows:

1. $v(A \wedge_t B) = v(A) \sqcap_t v(B)$;
2. $v(A \vee_t B) = v(A) \sqcup_t v(B)$;
3. $v(\sim_t A) = -_t v(A)$;
4. $v\left(A \wedge_f B\right) = v(A) \sqcup_f v(B)$;
5. $v\left(A \vee_f B\right) = v(A) \sqcap_f v(B)$;
6. $v\left(\sim_f A\right) = -_f v(A)$.

The relations $\models_t \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ and $\models_f \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ are defined by the following equivalences:

$$\Delta \models_t \Gamma \quad \text{iff} \quad \forall v \ \textstyle\bigsqcap_t \{v(A) \mid A \in \Delta\} \leq_t \bigsqcup_t \{v(A) \mid A \in \Gamma\};$$
$$\Delta \models_f \Gamma \quad \text{iff} \quad \forall v \ \textstyle\bigsqcup_f \{v(A) \mid A \in \Gamma\} \leq_f \bigsqcap_f \{v(A) \mid A \in \Delta\}.$$

We now define a separated 16-valued logic in the language $\mathcal{L}_{tf}$.

DEFINITION 16. The separated 16-valued logic B_{16} is the structure $\langle \mathbf{16}, \{x \in \mathbf{16} \mid x^t \text{ is non-empty}\}, \{x \in \mathbf{16} \mid x^f \text{ is non-empty}\}, \{-_t, \sqcap_t, \sqcup_t, -_f, \sqcup_f, \sqcap_f\}\rangle$. Moreover, for all sets of $\mathcal{L}_{tf}$-formulas Δ, Γ, semantic consequence relations $\models^+$ and $\models^-$ are defined in accordance with 1. and 2. of Definition 1.

OBSERVATION 17. B_{16} is refined.

Proof. It can easily be seen that in B_{16} the relations $\models^+$ and $\models^-$ are distinct, e.g. in view of the following counterexample: $(A \wedge_f B) \models^- A$ but $(A \wedge_f B) \not\models^+ A$ (since, e.g., $\boldsymbol{F} \sqcup_f \mathbf{FTB} = \mathbf{FB}$). ■

PROPOSITION 18. *The* 16*-valued propositional logic* B_{16} *is harmonious.*

Proof. Obviously, we may view the language $\mathcal{L}_{tf}$ as being based on a set of positive connectives $C^+ = \{\sim_t, \wedge_t, \vee_t\}$ and a set of negative connectives with matching arity $C^- = \{\sim_f, \wedge_f, \vee_f\}$, i.e., $C = \{\sim, \wedge, \vee\}$. Moreover, we already observed that the condition $f_{c_t} \neq f_{c_f}$ is satisfied for every $c \in C$. If A is an $\mathcal{L}_t$-formula, let A^f be the result of replacing every connective c_t in A by c_f. If Δ is a set of $\mathcal{L}_t$ formulas, let $\Delta^f = \{A^f \mid A \in \Delta\}$. It remains to be shown that in B_{16} for all sets of $\mathcal{L}_t$-formulas Δ, Γ,

$$(\dagger) \quad \Delta \models^+ \Gamma \text{ iff } \Delta^f \models^- \Gamma^f.$$

This follows from Lemma 4.2, Lemma 4.3, Theorem 4.4 and Theorem 4.7 in [28] for $\leq_t$ and the analogous versions of these statements for $\leq_f$. Lemma 4.3 says that for every $A, B \in \mathcal{L}_t$: $A \models_t B$ iff $\forall v(\mathbf{T} \in v(A) \Rightarrow \mathbf{T} \in v(B))$. According to Lemma 4.2, within language $\mathcal{L}_t$, the condition $\forall v(\mathbf{T} \in v(A) \Rightarrow \mathbf{T} \in v(B))$ is equivalent to $\forall v(\mathbf{B} \in v(A) \Rightarrow \mathbf{B} \in v(B))$. Thus, for every $A, B \in \mathcal{L}_t$: $A \models_t B$ iff $\forall v(\mathbf{T} \in v(A)$ or $\mathbf{B} \in v(A) \Rightarrow \mathbf{T} \in v(B)$ or $\mathbf{B} \in v(B))$. This means that $\models^+$ restricted to $\mathcal{L}_t$ is the same relation as $\models_t$ restricted to $\mathcal{L}_t$, and it is then axiomatized as first degree entailment (Theorems 4.4 and 4.7). Since for every $A, B \in \mathcal{L}_f$: $A \models_f B$ iff $\forall v(\mathbf{F} \in v(B) \Rightarrow \mathbf{F} \in v(A))$ iff $\forall v(\mathbf{B} \in v(B) \Rightarrow \mathbf{B} \in v(A))$, and the restriction of $\models^-$ to $\mathcal{L}_f$ (= the restriction of $\models_f$ to $\mathcal{L}_f$) is *also* axiomatized as first degree entailment, condition (†) is satisfied. ■

5 Harmony ad infinitum

The trilattice $SIXTEEN_3$ is an example of a *multilattice*, see [28], [34].

DEFINITION 19. An n-dimensional multilattice (or just n-lattice) is a structure $\mathcal{M}_n = (S, \leq_1, \ldots, \leq_n)$ such that S is a non-empty set and $\leq_1, \ldots, \leq_n$ are partial orders defined on S such that $(S, \leq_1), \ldots, (S, \leq_n)$ are pairwise distinct lattices.

More concretely, the structure $SIXTEEN_3$ is an example of a *Belnap-trilattice*, see [30]. Belnap-trilattices are obtained by iterated powerset-formation applied to the set **4** and by generalizing the definitions of a truth order and a falsity order on **16**. If X is a set, let $\mathcal{P}^1(X) := \mathcal{P}(X)$ and $\mathcal{P}^n(X) := \mathcal{P}(\mathcal{P}^{n-1}(X))$ for $1 < n$, $n \in \mathbb{N}$. We obtain an infinite collection of sets of generalized semantical values by considering $\mathcal{P}^n(\mathbf{4})$. Each of these sets can be equipped with relations $\leq_i$, $\leq_t$, and $\leq_f$ in a canonical way as in Definition 11 by defining for every $x, y \in \mathcal{P}^n(\mathbf{4})$ the sets x^t, x^{-t}, x^f and x^{-f} as follows:

$$x^t := \{y_0 \in x \mid (\exists y_1 \in y_0)(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\, T \in y_{n-1}\}$$

$$x^{-t} := \{y_0 \in x \mid \neg(\exists y_1 \in y_0)(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\, T \in y_{n-1}\}$$

$$x^f := \{y_0 \in x \mid (\exists y_1 \in y_0)(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\, F \in y_{n-1}\}$$

$$x^{-f} := \{y_0 \in x \mid \neg(\exists y_1 \in y_0)(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\, F \in y_{n-1}\}$$

Thus, $x^{-t} = x \setminus x^t$ and $x^{-f} = x \setminus x^f$. We say that x is t-positive (t-negative, f-positive, f-negative) iff x^t (x^{-t}, x^f, x^{-f}) is non-empty and we denote by $\mathcal{P}^n(\mathbf{4})^t$ ($\mathcal{P}^n(\mathbf{4})^{-t}$, $\mathcal{P}^n(\mathbf{4})^f$, $\mathcal{P}^n(\mathbf{4})^{-f}$) the set of all t-positive (resp. t-negative, f-positive, f-negative) elements of $\mathcal{P}^n(\mathbf{4})$.

DEFINITION 20. A Belnap-trilattice is a structure

$$\mathcal{M}_3^n := (\mathcal{P}^n(\mathbf{4}), \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f),$$

where $\sqcap_i$ ($\sqcap_t$, $\sqcap_f$) is the lattice meet and $\sqcup_i$ ($\sqcup_t$, $\sqcup_f$) is the lattice join with respect to the ordering $\leq_i$ ($\leq_t$, $\leq_f$) on $\mathcal{P}^n(\mathbf{4})$, $n \geq 1$.

Thus, $SIXTEEN_3$ ($= \mathcal{M}_3^1$) is the smallest Belnap-trilattice. Moreover, if unary operations $-_t$ and $-_f$ satisfying the conditions from Definition 13 exist on $\mathcal{P}^n(\mathbf{4})$, then we may consider Belnap-trilattices with t-inversion and f-inversion.

PROPOSITION 21. *If $\mathcal{M}_3^n$ is a Belnap-trilattice, then there exist operations of t-inversion and f-inversion on $\mathcal{P}^n(\mathbf{4})$.*

In [30] a canonical definition for such inversion operations is presented. Thus, we may without loss of generality confine our considerations to Belnap-trilattices with t-inversions and f-inversions.

We consider again the languages $\mathcal{L}_t$, $\mathcal{L}_f$, $\mathcal{L}_{tf}$ defined in the previous section. An n-valuation is a function v^n from the set of propositional variables into $\mathcal{P}^n(\mathbf{4})$. This function can be extended to an interpretation of arbitrary formulas in $\mathcal{P}^n(\mathbf{4})$ by Definition 15 (replacing v by v^n).

DEFINITION 22. The relations $\models_t^n \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ and $\models_f^n \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ are defined by the following equivalences:

$$\Delta \models_t^n \Gamma \quad \text{iff} \quad \forall v^n \ \bigsqcap_t\{v^n(A) \mid A \in \Delta\} \leq_t \bigsqcup_t\{v^n(A) \mid A \in \Gamma\};$$
$$\Delta \models_f^n \Gamma \quad \text{iff} \quad \forall v^n \ \bigsqcup_f\{v^n(A) \mid A \in \Gamma\} \leq_f \bigsqcap_f\{v^n(A) \mid A \in \Delta\}.$$

Semantically, the logic of a Belnap-trilattice $\mathcal{M}_3^n$ is the bi-consequence system $(\mathcal{L}_{tf}, \models_t^n, \models_f^n)$.

We now define an infinite chain of separated finitely-valued logics.

DEFINITION 23. Let $\sharp n$ be the cardinality of $\mathcal{P}^n(\mathbf{4})$. The $\sharp n$-valued logic $B_{\sharp n}$ is the structure $\langle \mathcal{P}^n(\mathbf{4}), \mathcal{D}^{n+}, \mathcal{D}^{n-}, \{-_t, \sqcap_t, \sqcup_t, -_f, \sqcup_f, \sqcap_f\}\rangle$, where $\mathcal{D}^{n+} := \{x \in \mathcal{P}^n(\mathbf{4}) \mid x \text{ is } t\text{-positive}\}$ and $\mathcal{D}^{n-} := \{x \in \mathcal{P}^n(\mathbf{4}) \mid x \text{ is } f\text{-positive}\}$. For every logic $B_{\sharp n}$, for all sets of $\mathcal{L}_{tf}$-formulas Δ, Γ, the semantic consequence relations $\models^{n+}$ and $\models^{n-}$ are defined in accordance with 1. and 2. of Definition 1.

OBSERVATION 24. For every $n \in \mathbb{N}$, the logic $B_{\sharp n}$ is refined.

Proof. Obviously, $B_{\sharp n}$ is separated. That in $B_{\sharp n}$ the relations $\models^{n+}$ and $\models^{n-}$ are distinct can again be seen by noticing that for every $n \in \mathbb{N}$, $(A \wedge_f B) \models^{n-} A$ but $(A \wedge_f B) \not\models^{n+} A$ (since for every $\mathcal{M}_3^n$ there may well exist $x, y \in \mathcal{P}^n(\mathbf{4})$, such that $x \sqcup_f y$ is t-positive, whereas either x or y is not). ■

PROPOSITION 25. *For every $n \in \mathbb{N}$, the logic $B_{\sharp n}$ is harmonious.*

Proof. Again, we regard $\mathcal{L}_{tf}$ as being based on $C^+ = \{\sim_t, \wedge_t, \vee_t\}$ and the set of connectives with matching arity $C^- = \{\sim_f, \wedge_f, \vee_f\}$, so that $C = \{\sim, \wedge, \vee\}$. Again, it can easily be seen that the condition $f_{c_t} \neq f_{c_f}$ is satisfied for every $c \in C$. We must show that in $B_{\sharp n}$ for all sets of $\mathcal{L}_t$-formulas Δ, Γ,

$$\Delta \models^{n+} \Gamma \text{ iff } \Delta^f \models^{n-} \Gamma^f.$$

We use the main result of [30], namely that for *every* $n \in \mathbb{N}$, the truth entailment relation $\models_t^n$ of $(\mathcal{L}_{tf}, \models_t^n, \models_f^n)$ restricted to $\mathcal{L}_t$ and the falsity entailment relation $\models_f^n$ of $(\mathcal{L}_{tf}, \models_t^n, \models_f^n)$ restricted to $\mathcal{L}_f$ can again both be axiomatized as first degree entailment. The proof systems differ only insofar, as every connective c_t is uniformly replaced by its negative counterpart c_f. Thus, given the completeness theorem for $\models_t^n$ and its counterpart for $\models_f^n$, it is enough to show that for all sets of $\mathcal{L}_t$-formulas Δ, Γ: $\Delta \models^{n+} \Gamma$ in $B_{\sharp n}$ iff $\Delta \models_t^n \Gamma$, and for all sets of $\mathcal{L}_f$-formulas Δ, Γ: $\Delta \models^{n-} \Gamma$ in $B_{\sharp n}$ iff $\Delta \models_f^n \Gamma$. But this follows from Corollary 17 in [30]: For any $A, B \in \mathcal{L}_t$:

$$A \models_t^n B \text{ iff } \ \forall v^n (x \in v^n(A)^t \Rightarrow x \in v^n(B)^t)$$

and the analogous result for falsity entailment: For any $A, B \in \mathcal{L}_f$:

$$A \models_f^n B \text{ iff } \ \forall v^n (x \in v^n(B)^f \Rightarrow x \in v^n(A)^f).$$

■

6 Some remarks on generalizing harmony

The point of departure for our considerations was the familiar and simple notion of an extensional n-valued propositional logic. We argued that the distinction between designated, antidesignated, and undesignated values and values that are both designated and antidesignated ought to be taken seriously. There is no reason to privilege designation over antidesignation when it comes to defining entailment. Hence we suggested a slightly more general notion of an n-valued logic. Whereas the set $\mathcal{D}^+$ of designated values determines a positive entailment relation $\models^+$, the set $\mathcal{D}^-$ of antidesignated values determines a negative entailment relation $\models^-$. Once we have two entailment relations, it is natural to consider two languages. This consideration then led us to defining the notion of a harmonious n-valued logic.

It has been observed quite a while ago already that one may consider n-valued logics in which the truth functions $f_\wedge$ and $f_\vee$ for conjunction and disjunction form a lattice on the underlying set of truth values, see, for example [24], [25]. Thus, given an n-valued propositional logic, one may wonder whether a lattice order can be defined from some given truth functions. However, in the light of the research on bilattices and trilattices, the direction of interest goes in the opposite direction. Given some natural partial orders on sets of semantical values, one may wonder whether these orderings form lattices on the underlying sets and thereby give rise to a conjunction (lattice meet), disjunction (lattice join), and hopefully also some sort of negation . In the bilattice $FOUR_2$, there is only one "logical" order, there referred to as the truth order. In a Belnap-trilattice $\mathcal{M}_3^n$, there are two logical orderings, the truth order $\leq_t$ and the falsity order $\leq_f$. From the bi-consequence system of any Belnap-trilattice we obtained a harmonious finitely-valued logic. But neither did we consider the following relation induced by the information order $\models^i$, defined as

$$\Delta \models_i^n \Gamma \quad \text{iff} \quad \forall v^n \ \bigsqcap_i \{v^n(A) \mid A \in \Delta\} \leq_i \bigsqcup_i \{v^n(A) \mid A \in \Gamma\},$$

nor did we consider additional orderings or other sets of truth values with more than two lattice orderings on them. This would, however, be an interesting direction to pursue, if one aims at finding examples of many-valued logics displaying a more general form of harmony. Thus, generalizations of the notion of a harmonious n-valued propositional logic can be obtained by replacing the set $\{\mathcal{D}^+, \mathcal{D}^-\}$ of distinguished subsets of the set of values $\mathcal{V}$ by a set $\{\mathcal{D}_1, \ldots, \mathcal{D}_n\}$ of distinguished subsets of $\mathcal{V}$. Moreover, the association of entailment relations with the distinguished subsets of $\mathcal{V}$ may vary.

A relation $\models \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ is a Tarski-Scott multiple conclusion consequence relation iff it satisfies the following conditions:

1. For every $\Delta \subseteq \mathcal{L}$, $\Delta \models \Delta$ (reflexivity);
2. If $\Delta \models \Gamma \cup \{A\}$ and $\{A\} \cup \Theta \models \Sigma$, then $\Delta \cup \Theta \models \Gamma \cup \Sigma$ (transitivity);
3. If $\Delta \subseteq \Theta$, $\Gamma \subseteq \Sigma$, and $\Delta \models \Gamma$, then $\Theta \models \Sigma$ (monotony).

A Tarski-Scott multiple conclusion consequence relation is said to be structural, if it is closed under substitution.

In an n-valued logic which is refined in the sense of Definition 2, in addition to $\models^+$ and $\models^-$ several other, possibly interesting semantical relations can be defined, though not all of them turn out to be Tarski-Scott multiple conclusion entailment relations. In order to refer to these relations in a compact way, we need a more systematic notation. Let $+$ stand for designated values, $-$ for antidesignated values, u for neither designated nor antidesignated values, and b for values that are both designated and antidesignated. Let $\overline{+}$, $\overline{-}$, $\overline{\mathsf{u}}$, and $\overline{\mathsf{b}}$ stand for the respective complements. Moreover, let $\Leftarrow$ indicate preservation from the conclusions to the premises, $\Rightarrow$ preservation from the premises to the conclusions, and $\Leftrightarrow$ preservation in both directions. Then $\models^+$ is denoted as $\models^{+\Rightarrow+}$ and $\models^-$ as $\models^{-\Leftarrow-}$. The relation $\models$ defined (for a given n-valued logic) by requiring that $\Delta \models \Gamma$ iff both $\Delta \models^+ \Gamma$ and $\Delta \models^- \Gamma$ (cf. the proof of Observation 9) is denoted as $\models^{(+\Rightarrow+,-\Leftarrow-)}$, and the relation defined by requiring that $\Delta \models \Gamma$ iff ($\Delta \models^+ \Gamma$ or $\Delta \models^- \Gamma$) is denoted as $\models^{(+\Rightarrow+|-\Leftarrow-)}$. We may consider in addition to $\models^+$ and $\models^-$ for example the following "semantic consequence" relations:

- $\models^{+\Leftarrow+}$;
- $\models^{-\Rightarrow-}$;

- $\models^{\mathsf{u}\Rightarrow\mathsf{u}}$; $\models^{\mathsf{u}\Leftarrow\mathsf{u}}$;
- $\models^{\mathsf{b}\Rightarrow\mathsf{b}}$; $\models^{\mathsf{b}\Leftarrow\mathsf{b}}$;
- $\models^{\overline{+}\Rightarrow\overline{+}}$; $\models^{\overline{+}\Leftarrow\overline{+}}$;
- $\models^{\overline{-}\Rightarrow\overline{-}}$; $\models^{\overline{-}\Leftarrow\overline{-}}$;
- $\models^{\overline{\mathsf{u}}\Rightarrow\overline{\mathsf{u}}}$; $\models^{\overline{\mathsf{u}}\Leftarrow\overline{\mathsf{u}}}$;
- $\models^{\overline{\mathsf{b}}\Rightarrow\overline{\mathsf{b}}}$; $\models^{\overline{\mathsf{b}}\Leftarrow\overline{\mathsf{b}}}$;
- $\models^{\bullet\Rightarrow\circ}$; $\models^{\bullet\Leftarrow\circ}$;
  where $\bullet, \circ \in \{+, -, \mathsf{u}, \mathsf{b}, \overline{+}, \overline{-}, \overline{\mathsf{u}}, \overline{\mathsf{b}}\}$ and $\bullet \neq \circ$;
- $\models^{(\bullet\Rightarrow\circ,\blacklozenge\Leftarrow\lozenge)}$; $\models^{(\bullet\Leftarrow\circ,\blacklozenge\Rightarrow\lozenge)}$;
  where $\bullet, \circ, \blacklozenge, \lozenge \in \{+, -, \mathsf{u}, \mathsf{b}, \overline{+}, \overline{-}, \overline{\mathsf{u}}, \overline{\mathsf{b}}\}$ and $\bullet \neq \overline{\blacklozenge}$ or $\circ \neq \overline{\lozenge}$;
- $\models^{(\bullet\Rightarrow\circ,\blacklozenge\Rightarrow\lozenge)}$; $\models^{(\bullet\Leftarrow\circ,\blacklozenge\Leftarrow\lozenge)}$;
  where $\bullet, \circ, \blacklozenge, \lozenge \in \{+, -, \mathsf{u}, \mathsf{b}, \overline{+}, \overline{-}, \overline{\mathsf{u}}, \overline{\mathsf{b}}\}$ and $\bullet \neq \blacklozenge$ or $\circ \neq \lozenge$;
- $\models^{(\bullet\Rightarrow\circ|\blacklozenge\Leftarrow\lozenge)}$; $\models^{(\bullet\Leftarrow\circ|\blacklozenge\Rightarrow\lozenge)}$;
  where $\bullet, \circ, \blacklozenge, \lozenge \in \{+, -, \mathsf{u}, \mathsf{b}, \overline{+}, \overline{-}, \overline{\mathsf{u}}, \overline{\mathsf{b}}\}$ and $\bullet \neq \overline{\blacklozenge}$ or $\circ \neq \overline{\lozenge}$;
- $\models^{(\bullet\Rightarrow\circ|\blacklozenge\Rightarrow\lozenge)}$; $\models^{(\bullet\Leftarrow\circ|\blacklozenge\Leftarrow\lozenge)}$;
  where $\bullet, \circ, \blacklozenge, \lozenge \in \{+, -, \mathsf{u}, \mathsf{b}, \overline{+}, \overline{-}, \overline{\mathsf{u}}, \overline{\mathsf{b}}\}$ and $\bullet \neq \blacklozenge$ or $\circ \neq \lozenge$.

Since consequence relations normally are not required to be symmetric, relations like $\models^{+\Leftrightarrow+}$ are, perhaps, not of primary interest. But the relation $\models^{(+\Leftrightarrow+,\mathsf{u}\Rightarrow\overline{-})} = \models^{+\Leftrightarrow+} \cap \models^{\mathsf{u}\Rightarrow\overline{-}}$, for example, might be of some interest. Investigating and applying such non-standard semantic consequence relations is not as exotic as it might seem at first sight, and, indeed, some such relations have been considered in the literature. We already noted that $\models^{(+\Rightarrow+,-\Leftarrow-)}$ is dealt with in [10], and there are other examples.

EXAMPLE 26. G. Malinowski [17], [18], [19], emphasizing the distinction between accepted and rejected propositions, draws the distinction between designated and antidesignated values and uses it to generalize Tarski's notion of a consequence operation to the notion of a quasi-consequence operation (or just q-consequence operation). Also, single-conclusion q-consequence relations are defined; they relate not antidesignated assumptions to designated (single) conclusions. In our notation, multiple-conclusion q-consequence is the relation $\models^{\overline{-}\Rightarrow+}$.

EXAMPLE 27. Another non-standard consequence relation has been presented in [7] and is there said *not* to be "overly outlandish or inconceivable", although it fails to be a Tarski-Scott multiple-conclusion consequence relation. In our notation, the "tonk-consequence" relation of [7] is the relation $\models^{(+\Rightarrow+|-\Leftarrow-)}$ on the set **4** with $\mathcal{D}^+ = \{\mathbf{T}, \mathbf{B}\}$ and $\mathcal{D}^- = \{\mathbf{F}, \mathbf{B}\}$. Since the logic is not transitive, sound truth tables for Prior's connective `tonk` are available such that this addition of `tonk` does not have a trivializing effect (but see also [35]).

EXAMPLE 28. Formula-to-formula q-consequence, though not under this name, is also among the varieties of semantic consequence considered in Chapters 3 and 4 of [33]. The types of consequence relations presented by Thijsse include the relations which in our notation for the multiple conclusion case are denoted as follows: $\models^{+\Rightarrow+}$, $\models^{\overline{-}\Rightarrow+}$, $\models^{\overline{-}\Rightarrow\overline{-}}$, $\models^{+\Rightarrow\overline{-}}$. Thijsse remarks that besides the familiar $\models^{+\Rightarrow+}$ the relation $\models^{\overline{-}\Rightarrow\overline{-}}$ "turns out to be interesting, both in theory and application".

Yet another way of defining a generalized notion of semantic consequence can be found in [5]. The two types of relation $\models^{+\Rightarrow+}$ and $\models^{-\Rightarrow-}$ are there merged into a single *four-place* bi-consequence relation. Note also that n-place semantic sequents for n-valued logics have been considered by Schröter [26], see also [14], [15].

After these preparatory remarks, we are in a position to define a generalized notion of a harmonious n-valued propositional logic.

DEFINITION 29. Let $\mathcal{L}$ be a language in a denumerable set of sentence letters and a finite non-empty set of finitary connectives C. An n-valued propositional logic is a structure

$$\langle \mathcal{V}, \mathcal{D}_1, \ldots, \mathcal{D}_k, \{f_c : c \in C\} \rangle$$

where $\mathcal{V}$ is a non-empty set of cardinality n $(2 \leq n)$, $2 \leq k$, every $\mathcal{D}_i$ $(1 \leq i \leq k)$ is a non-empty proper subset of $\mathcal{V}$, the sets $\mathcal{D}_i$ are pairwise distinct, and every f_c is a function on $\mathcal{V}$ with the same arity as c. The sets $\mathcal{D}_i$ are called distinguished sets. A valuation v is inductively extended to a function from the set of all $\mathcal{L}$-formulas into $\mathcal{V}$ by setting: $v(c(A_1, \ldots, A_m)) = f_c(v(A_1), \ldots, v(A_m))$, for every m-place $c \in C$. For every set $\mathcal{D}_i$, two semantic consequence relation $\models_i^{\Rightarrow}$ and $\models_i^{\Leftarrow}$ are defined as follows:

1. $\Delta \models_i^{\Rightarrow} \Gamma$ iff for every valuation function v: (if for every $A \in \Delta$, $v(A) \in \mathcal{D}_i$, then $v(B) \in \mathcal{D}_i$ for some $B \in \Gamma$);
2. $\Delta \models_i^{\Leftarrow} \Gamma$ iff for every valuation function v: (if for every $A \in \Gamma$, $v(A) \in \mathcal{D}_i$, then $v(B) \in \mathcal{D}_i$ for some $B \in \Delta$).

Obviously, the relations $\models_i^{\Rightarrow}$ and $\models_i^{\Leftarrow}$ are inverses of each other.

DEFINITION 30. Let $\Lambda = \langle \mathcal{V}, \mathcal{D}_1, \ldots, \mathcal{D}_k, \{f_c : c \in C\} \rangle$ be an n-valued logic. Λ is said to be separated, if for every $\mathcal{D}_i$, there exists no $\mathcal{D}_j$ such that $i \neq j$ and $\mathcal{V} \setminus \mathcal{D}_i = \mathcal{D}_j$. Λ is said to be refined, if it is separated and the relations $\models_i^{\Rightarrow}$ are pairwise distinct (and hence also the relations $\models_i^{\circ}$ with $\circ \in \{\Rightarrow, \Leftarrow\}$ are distinct).

DEFINITION 31. Let C be a finite non-empty set of finitary connectives, and let $\mathcal{L}_i$ be the language based on $C_i = \{c_i \mid c \in C\}$ for some $k \in \mathbb{N}$ such that $2 \leq i \leq k$. If $i \neq j$, $2 \leq j \leq k$ and if A is an $\mathcal{L}_i$-formula, then let A_j be the result of replacing every connective c_i in A by c_j. If Δ is a set of $\mathcal{L}_i$ formulas, let $\Delta_j = \{A_j \mid A \in \Delta\}$. If the language $\mathcal{L}$ of a refined n-valued logic Λ with k distinguished sets is based on the set $\bigcup_{i \leq k} C_i$, then Λ is said to be *harmonious* iff (i) for every i, j with $i \neq j$ and all sets of $\mathcal{L}_i$-formulas Δ, Γ the following holds: (i) $\Delta \models_i^{\Rightarrow} \Gamma$ iff $\Delta_j \models_j^{\Leftarrow} \Gamma_j$, (ii) $\Delta \models_i^{\Leftarrow} \Gamma$ iff $\Delta_j \models_j^{\Rightarrow} \Gamma_j$, and (iii) for every $c \in C$, $f_{c_i} \neq f_{c_j}$.

The logics $B_{\sharp n}$ are harmonious in this generalized sense. We may set $k = 2$, $\mathcal{D}_1 = \mathcal{D}^+$, $\mathcal{D}_2 = \mathcal{D}^-$, $C_1 = \{\sim_t, \wedge_t, \vee_t\}$, $C_2 = \{\sim_f, \wedge_f, \vee_f\}$, $\models_1^{\Rightarrow} = \models^+$, $\models_1^{\Leftarrow} = (\models^+)^{-1}$ (the inverse of $\models^+$), $\models_2^{\Leftarrow} = \models^-$, and $\models_2^{\Rightarrow} = (\models^-)^{-1}$.

Another obvious generalization of the present considerations is giving up the restriction to finitely many values.

We end this paper with the following question: Are there natural examples of harmonious n-valued logics $\langle \mathcal{V}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \{f_c : c \in C\} \rangle$ with *three* distinguished sets of semantical values?

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